Comment on **High-overtone normal modes of Schwarzschild black holes**

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Abstract

The validity of the JWKB approximation as applied to black hole normal modes is discussed as a possible source of the discrepancy between the quasinormal frequency values computed by Guinn, Will, Kojima, and Schutz in their recent letter, and those computed by the method of continued fractions. For Regge-Wheeler-like potentials and large frequency magnitudes, an analytic JWKB approximation is suggested that can be expressed in terms of complete elliptic integrals.

1 Introduction

In a recent Letter, Guinn et al. [1] have computed values for the large overtone Schwarzschild quasinormal frequencies that differ both quantitatively and qualitatively from those published by this author five years previously. While the quasinormal frequencies Guinn et al. compute for the Regge-Wheeler potential via the second order JWKB approximation agree closely with Chandrasekhar and Detweiler's numerical integration [2] and our continued fraction results [3] at the fundamental and first overtone resonances, they systematically diverge from the continued fraction values as the overtone index increases. At asymptotically large overtone indices, the real parts of the quasinormal frequencies as computed by the JWKB method approach zero, while the continued fraction result places them on an asymptote parallel to, but bounded away from, the imaginary frequency axis (Guinn et al. Figure 3). As part of this systematic discrepancy, the JWKB method does not suggest the possible existence of a quasinormal mode at the algebraically special frequency. The authors conclude "The possibility that Leaver's calculation gives wrong values for the real parts of the frequencies must be considered... on the other hand, we have looked at this and not found any obvious flaws. These questions urgently need to be clarified, because of the potentially wide applicability of the WKB method to other problems..."

Details of the continued fraction method are given in references [3, 4, 5, 6]. The values originally obtained for the lowest six $l = 2$ quasinormal frequencies were confirmed by computing the Wronskian of the quasinormal mode wavefunction with the purely-outgoing-at-infinity wavefunction in reference [7], and more recently by Majumdar and Panchepakasan [8] by a matrix-determinant method, and by Andersson [9] via an elegant numerical integration. These four different methods give the same results to the precision computed; the lowest six quasinormal frequency values should not be in question. Numerical techniques used to evaluate continued fractions are well established [10, 11, 12], and are generally quite robust. Further discussion of the convergence, validity, and accuracy of the method will appear in reference [13].

2 The JWKB Approximation and its Validity

The discussion accompanying the first nine equations of Guinn et al.'s letter, together with their Figures 1 and 2 and Bender and Orzag's [14] eq. 10.1.17, is sufficient to describe the JWKB problem. It must be recalled that the JWKB series is almost always asymptotic [14]; prior to the efforts of Schutz, Will, and their coworkers [1, 15, 16, 17, 18, 19], the method does not appear to have been applied to the potential barrier scattering - quasinormal mode eigenvalue problem. Since the Regge-Wheeler potential changes appreciably over the effective quasinormal wavelength near the potential's peak, the validity of the JWKB approximation is open to question. We suggest Schutz and Will's original assertion [15] that the approach "will be powerful because the JWKB approximation can be carried to higher orders, either as a means to improve the accuracy or as a means to establish errors explicitly" has yet to be established: it at least requires the tabular comparison of values from successively higher JWKB approximations (third order minimum) against increasing overtone index using a consistent potential approximation and integration scheme throughout, and against known reliable values of fundamental and lowest overtone frequencies (e.g. those for which references [2] and [17] agree, or those listed in Table I of reference [7]).

Guinn et al. do not cite such a systematic comparison; their belief in the validity of their results appears to stem from the 3 - 4% agreement between their first order JWKB results and the continued fraction/numerical integration results for overtone indices $n = 0, 1$, and 2, and from the relatively close agreement between their first and second order approximations: 1% at $n = 10$ and 0.04% at $n = 60$. Their implicit assumption is that the asymptotic JWKB series can then be optimally *and* accurately truncated after the second term for all values of the frequency.

We see no basis for this assumption. Guinn et al.'s equations 5 (with Bender and Orzag's 10.1.17) indicate that the eikonal term S_0 is the dominant contributor to the S series simply because the magnitude of its residue ρ at the pole $r = 2M$ becomes arbitrarily large as the overtone index increases. None of the transport terms $S_2, S_4 \ldots$ contribute at all to this residue [1]. Convergence of the JWKB series depends on the convergence of the successive transport terms on contour C_3, about which the numerical contribution of the eikonal term S_0 and (relatively small) first transport term S_2 alone tells us nothing. For this problem the relative accuracy of the JWKB approximation can at best be self-assessed only when there exists a decreasing subsequence of the transport terms at the frequency value of interest. Otherwise it is not clear that even the first transport term S_2 makes a valid contribution.

3 The quasinormal mode at the algebraically special frequency

The algebraically special gravitational perturbations, one for each multipole, occur at the purely imaginary frequencies $\omega_{\rm as} = -il(l-1)(l+1)(l+2)/12M$. This case was not investigated carefully at the time we prepared our original quasinormal mode article (save to note the continued fraction did not converge there), but when preparing reference [7] the Coulomb wavefunction expansion therein described was used to investigate the asymptotic behavior of the wavefunction that was ingoing at the horizon in a small frequency neighborhood encircling the algebraically special value. We were satisfied that wavefunction did indeed satisfy the outgoing quasinormal mode boundary condition at infinity, at the algebraically special frequency. Note the distinction between "quasinormal mode" and "algebraically special mode." These terms refer to actual functional solutions to the Regge-Wheeler equation. Our claim is that there exists a quasinormal mode at the algebraically special frequency. Obviously, the quasinormal and algebraically special modes cannot be the same; they are just one way to express the Regge-Wheeler equation's two independent solutions at the algebraically special frequency.

4 Asymptotic overtones above the algebraically special frequency

It should be noted the continued fraction method can compute overtone values for n much larger than 60. Our original computations were halted at that particular value only because we felt we had by then sufficiently established the general nature of the asymptotic Schwarzschild quasinormal frequency distribution. The asymptotic behavior Guinn, Will et al. obtain for the high-overtone quasinormal frequencies of the Regge-Wheeler potential, in addition to differing qualitatively from the continued fraction result, also differs qualitatively from the high-overtone quasinormal frequency behavior of the only other analytic potential whose quasinormal frequency values are known (at least to us), namely, the Eckart potential investigated by Blome and Mashhoon [21]. The Eckart potential is

$$V(x) = V_0 e^{2\mu} - V_0[\tanh(\alpha x + \mu) - \tanh\mu]^2 \cosh^2\mu \quad , \tag{1}$$

and reduces to the $\mathrm{sech}^2(\alpha x)$ potential when $\mu = 0$. Blome and Mashhoon give the quasinormal frequencies of this potential as

$$\omega = \pm a(1+\Delta) - ib(1-\Delta) \tag{2}$$

where

$$\begin{aligned} a &= V_0^{1/2}\cosh\mu[1 - \alpha^2/4V_0\cosh^2\mu]^{1/2} \quad , & (3)\\ b &= \alpha(n+1/2) \quad , & (4)\\ \Delta &= \frac{V_0 \sinh 2\mu}{2(a^2+b^2)} \quad . & (5) \end{aligned}$$

These quasinormal frequencies have the same qualitative asymptotic distribution as was obtained by the continued fraction method for the Regge-Wheeler potential: the asymptote parallels the imaginary ω axis and is bounded away from it.

Interestingly, when applied to the Eckart potential the JWKB approximation may (with a thusfar totally unjustified bit of imagination) be used to *exactly* duplicate the known analytic quasinormal frequency values. Following the prescription of Guinn, et al.'s letter, it is straightforward to show that the first order JWKB values for the Eckart quasinormal frequencies are given by eqs. (2) - (5) but with a replaced by $a_0 = V_0^{1/2} \cosh\mu$. The second order result may be expressed similarly, but with a replaced by $a_2 = V_0^{1/2} \cosh\mu[1 - \alpha^2/(8V_0 \cosh^2\mu)]$. Note that a_0 and a_2 form the first two terms of the Taylor's expansion of a in the quantity $\alpha^2/(4V_0 \cosh^2\mu)$, which is a normalized measure of the curvature at the peak of the potential.

This may be fortuitous, and we have not investigated JWKB orders beyond the second. If the pattern holds it would imply the JWKB series for this potential is uniformly convergent, rather than asymptotic, a conclusion that certainly cannot be drawn from analysis of only the first two terms. Although there is no reason to suppose the Regge-Wheeler quasinormal frequencies will be as simply related to their JWKB approximations as their Eckart potential counterparts appear to be, something may be gained in attempting a similarly analytic result.

5 An analytic JWKB approach for Regge-Wheeler-like potentials at large frequency magnitudes

A key consideration is that when expressed as functions of r, the integrands of each of the JWKB integrals (Guinn et al.'s eqs. 4 and 5) may be expressed as the product of the ratios of two polynomials and the square root of a quartic. This makes them elliptic integrals [22]; that the endpoints r_3 and r_4 are roots of the quartic makes them complete. Such integrals may be expressed in closed analytic form, at least as $|\rho| \to \infty$, provided one can find simple expressions for the endpoints.

To this end one can refine Mashhoon's approach [21], which originally was to replace the Regge-Wheeler potential with a model potential that was both tractable to analysis, and whose quasinormal frequencies could be expected to have some properties in common with those of the Regge-Wheeler potential. Mashhoon adjusted the model potential's peak amplitude and curvature to match those of the Regge-Wheeler potential. He obtained good approximations to the lowest quasinormal frequencies, and a hint as to the nature of their asymptotic distribution.

It is precisely that asymptotic distribution we now wish to analyze. Not incidentally, Mashhoon's potentials also closely approximate the Regge-Wheeler potential's JWKB turning points near the fundamental frequency; the modification that suggests itself is to replace the Regge-Wheeler potential with a model potential that is both tractable to exact JWKB analysis, and which closely approximates the Regge-Wheeler potential on the JWKB integration contours for frequency values in the asymptotic regime. The model potential's turning points must be both exactly solvable and asymptotically approach those of the Regge-Wheeler potential in this region. To construct this model potential we follow Guinn et al. and expand the turning points of the Regge-Wheeler potential in inverse powers of $\sqrt{\rho}$, finding

$$r_i \sim \epsilon^{1/4}(-\rho^2)^{-1/4}\left[1-\tfrac{1}{4}(\epsilon+\lambda)\epsilon^{-3/4}(-\rho^2)^{-1/4}+O(\rho^{-1})\right] \quad , \tag{6}$$

where $\lambda = l(l+1)$ and the field spin parameter ϵ is $+3$ for gravitational perturbations. The four different turning points are obtained by taking the four different fourth roots of $-\rho^2$, taking the same fourth root at each occurrence in (6). We are interested in large magnitudes $|\rho|$ lying near the negative real axis; to the same order in ρ^{-1} the turning points of the Regge-Wheeler potential are shared by the model potential

$$V_{\text{model}} = -Q_{\text{model}} - \rho^2 = r^{-4}\left[\frac{(\epsilon+\lambda)^2}{4\epsilon}r^2 - (\epsilon+\lambda)r + \epsilon\right] \quad , \tag{7}$$

whose exact turning points may be found by simple factoring. The two potentials differ only in the coefficient of r^{-2}, which for the Regge-Wheeler potential is simply λ. Figure 1 of reference [3] suggests the asymptotic quasinormal frequency values are not strong functions of that parameter. The model potential closely approximates the Regge-Wheeler potential for small magnitudes of r, and it remains only to define a JWKB integration contour over which this condition holds.

Two possibilities suggest themselves. The first is the contour C_3 used by Guinn et al. (see their Figure 2.), provided care is taken to explicitly separate the contribution from the pole at $r = 0$. The second is a deformation of Guinn et al.'s contour C_0 and the branch cut it encircles, so that they run directly from turning point r_4 to a small distance from the origin, loop around the other branch cut to r_2 and back to the origin, loop around the pole at $r = 2M$ and back to the origin, then to their terminus at r_3. In both cases it is important the actual Regge-Wheeler potential be used to evaluate the integral around the pole at $r = 2M$, which lies outside the region the model potential is valid. Symmetries in the integrands may be exploited on both contours, although the presence of the extra branch cut inside the second (between r_2 and the origin) may introduce subtleties.

With these contours and turning points, the JWKB integrals of the Q_{model} (eq. 7) can be expressed in terms of complete elliptical integrals of analytic arguments, and in the limit of large ρ, eventually in closed analytic form. As the quasi-numeric asymptotic expression (equation 11 of their letter) arrived at by Guinn et al. may be more easily and accurately obtained by curve fitting continued fraction results, the JWKB method's greatest contribution (if it is valid for large overtones at all) will no doubt be realized through its potential to yield such fully analytic forms.

6 Acknowledgements and Conclusion

As validity conditions for the application of the JWKB approximation to black hole quasinormal mode problems have yet to be established, it is not surprising there should be discrepancies between JWKB results and those obtained by other methods at this time. Considerable work remains to be done. We received a copy of Nils Andersson's PhD dissertation [9] while this manuscript was in preparation. Among his findings is that the Zerilli potential yields better phase integral quasinormal frequency results than does Regge-Wheeler. A. Anderson and R.H. Price's [23] recent intertwining analysis may be useful in clarifying this issue.

Last, we are grateful to Professor Will for his invitation (personal communication) to respond to the issues he and his colleagues raised[1]. Professor Will also kindly sent notes from a presentation made by Hans-Peter Nollert at the Sixth Marcel Grossman Meeting (1991) while the present manuscript was under review. Nollert reports deriving a series expansion for the Wronskian of the ingoing and outgoing solutions to the Regge-Wheeler equation which is convergent for all frequencies. The quasinormal frequencies are just the zeroes of this Wronskian; Nollert plots their values for overtone indices up to $n \sim 2000$, and presents a simple formula for their asymptotic distribution. His results agree with those obtained from the continued fraction method, except (apparently) at the algebraically special frequencies. More important is the possibility his method might also provide values for the frequency derivative of the Wronskian at large overtone quasinormal frequencies, and thus allow completion of the normal mode expansion of the Green's function begun in reference [7]. We thank Drs. Andersson and Nollert for making their work available, and encourage timely publication of their results.

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