Remarks on the continued-fraction method for computing black-hole quasinormal frequencies and modes.

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INTRODUCTION

The continued-fraction method affords a stable, accurate, and convergent technique for computing quasinormal frequencies and modes. A review seems appropriate however, particularly in view of some apparent uncertainty concerning the method and conflicting JWKB results for the quasinormal frequencies. As instances, Anderson and Price [1] assert "Three term recursion relations can be treated by continued-fraction methods. With such a method, Leaver showed that the quasinormal frequencies are those complex values of ω for which the continued-fractions converge. He then uses the convergence of the continued-fractions as the basis for a precise and stable computation scheme for the quasinormal frequencies," while Guinn, Will, Kojima, and Schutz [2] suggest "The possibility that Leaver's calculation gives wrong values for the real parts of the frequencies must be considered, especially in view of the considerable delicacy of the numerical techniques he used to evaluate his continued-fractions; on the other hand, we have looked at this and not found any obvious flaws." Possible weaknesses in the JWKB analysis of Guinn *et al.* are discussed elsewhere [3]; here we take the opportunity to clarify the Anderson-Price statement concerning convergence, and other important continued-fraction issues.

THE CONTINUED-FRACTION METHOD FOR SCHWARZSCHILD QUASINORMAL FREQUENCIES AND MODES

Derivation and convergence

In units where $c = G = 2M = 1$ and with an $\exp(-i\omega t)$ time dependence, the Regge-Wheeler equation is

$$r(r-1)\psi_{,rr} + \psi_{,r} + \left[\frac{\omega^2 r^3}{r-1} - l(l+1) + \frac{\epsilon}{r}\right]\psi = 0 \quad , \tag{1}$$

where the field spin parameter $\epsilon = -1, 0, 3$ for scalar, electromagnetic, and gravitational perturbations. The solution that is in-going at the event horizon may be written (cf. Ref. [4])

$$\psi_l = (r-1)^{\rho} r^{-2\rho} e^{-\rho(r-1)} \sum_{n=0}^{\infty} a_n (1 - 1/r)^n \quad , \tag{2}$$

where the frequency parameter $\rho = -i\omega$ and the expansion coefficients a_n are defined by the three-term recurrence relation

$$\alpha_0 a_1 + \beta_0 a_0 = 0 \tag{3}$$

$$\alpha_n a_{n+1} + \beta_n a_n + \gamma_n a_{n-1} = 0 \quad n = 1, 2, \dots \ . \tag{4}$$

The recurrence coefficients α_n, β_n γ_n are explicit functions of the frequency ρ, multipole moment l, and field spin parameter ϵ:

$$\begin{aligned} \alpha_n &= n^2 + (2\rho + 2)n + 2\rho + 1 \quad , \\ \beta_n &= -[2n^2 + (8\rho + 2)n + 8\rho^2 + 4\rho + l(l+1) - \epsilon] \\ \gamma_n &= n^2 + 4\rho n + 4\rho^2 - \epsilon - 1 \quad . \end{aligned} \tag{5}$$

The quasinormal frequencies are then those complex values of ρ for which the *series* in Eq. (2) converges uniformly as $r \to \infty$. The convergence of this series is a separate issue from the convergence of the *continued-fraction*

$$F(\rho) = \frac{-\gamma_1}{\beta_1 -} \ \frac{\alpha_1 \gamma_2}{\beta_2 -} \ \frac{\alpha_2 \gamma_3}{\beta_3 -} \cdots \ , \tag{6}$$

which is an analytic function of the frequency ρ, and is empirically found to converge for all ρ that are not purely negative real. (This restriction is related to the absence of a minimal solution to recurrence equations (4) when ρ is negative real. See Refs. [4] Eq. (9), and [5] Eqs. (42)-(46).)

Now, when ρ is a quasinormal eigenfrequency ρ_q, the sequence of expansion coefficients $\{a_n(\rho_q); n = 0, 1, 2, \dots\}$ is the minimal solution to recurrence relation (4), and the ratio of the first two expansion coefficients is equal to the value of this continued-fraction [5, 6]:

$$a_1(\rho_q)/a_0(\rho_q) = F(\rho_q) \quad . \tag{7}$$

Since this ratio is also given by Eq. (3) for any ρ, we then have the equation

$$F(\rho_q) = -\beta_0(\rho_q)/\alpha_0(\rho_q) \quad , \tag{8}$$

which holds whenever ρ is a quasinormal frequency. However, when ρ is *not* a quasinormal frequency (and also not purely negative real), the continued-fraction *still* converges and the expression

$$\beta_0(\rho)/\alpha_0(\rho) + F(\rho) \tag{9}$$

is an analytic function of ρ whose *zeros* are the quasinormal frequencies. Being analytic, expression (9) makes an ideal target for a numerical root search.

A variety of methods are available. The MINPACK [7] library's nonlinear root search routine HYBRD can handle multiple equations and unknowns, which is particularly useful for Kerr black-holes where the eigenfrequency and angular separation constant are simultaneously sought. HYBRD works well in all cases, but for Schwarzschild and Reissner-Nordström black-holes the eigenfrequency is the only unknown and analyticity may be exploited by single-variable techniques such as Muller's method [8], used in routines such as the International Mathematical and Scientific Library's ZANLYT [9]. However, it must be emphasized that precision and initial step size control in the root-search routine are essential: in our double precision calculations we use an initial step size between 10^{-6} and 10^{-8}. The standard MINPACK distribution includes both single and double precision versions of all routines, and HYBRD affords fine control over step size, thus making it the nonlinear root-search routine of our personal choice. (We also obtain identical quasinormal frequency values using a double precision version of ZANLYT, after modifying the code to allow step-size control.) The continued-fraction itself may be stably evaluated by either of the algorithms given in Refs. [8] and [10] (also discussed in [6]); we have used them both with essentially identical results.

Stability and accuracy

An interesting convergence aspect of series (2) itself is that when ρ is an eigenfrequency $\{\rho_q; q = 0, 1, 2, \dots\}$ the largest of the expansion coefficients a_n is approximately a_q, so that for all but the fundamental $q = 0$ mode, the series coefficients form an increasing sequence between the first term a_0 and approximately a_q; uniform convergence does not set in until after this point. Consequently, it is (approximately) the qth inversion of the continued-fraction expression (9) that provides the most stable function for searching for the qth root, i.e., the qth root is easiest to find numerically, starting with the worst initial guess and largest initial step size, if the continued-fraction is first inverted approximately q times [(Ref. [4] Eq. (14)]. However, the qth quasinormal frequency ρ_q remains a solution to (9) regardless of the number of inversions, as can be verified by starting the root search close enough to ρ_q with a small enough step size.

Regarding accuracy, the selected values for the fundamental and first 59 overtone quasinormal frequencies listed in Ref. [4] were obtained by numerical solution of Eq. (8) on a computer with a 27-bit floating-point mantissa. This corresponds to a maximum of seven decimal digits. Note the precision is that of the complete complex quantity; the real parts of the high overtone frequencies, because of their hundred-fold magnitude difference from the imaginary parts, possess a few significant figures less than the imaginary parts. Use of a sequence convergence acceleration routine limited the number of approximants needed to compute the highest overtone values to fewer than 2000, although this speed optimization is not required on modern workstations and introduced some imprecision of its own. The continued-fraction was declared to have converged when the relative contribution of the last computed approximant became less than one part in 10^7, and the root-search acceptance criterion was one part in 10^5.

We have recently rerun the program on a DECstation 3100 workstation with 53-bit floating-point mantissa, without the convergence accelerator. The continued-fraction and root-search tolerances were set to 10^{-13} and 10^{-11} respectively, yielding frequency values to ten or twelve significant figures. The values of the lowest six $l = 2$ quasinormal frequencies listed in Table I of Ref. [4] remained unchanged, the seventh and eighth changed in the seventh and sixth significant figures, and values ten, eleven, and twelve remained unchanged. Frequency values for overtone indices $q = 19$ through 59 changed in the fifth significant figure. The accuracy for $l = 3$ was similar, save for those frequencies bracketing the algebraically special value, which erred in the fourth significant digit. They should read $(-0.03795, -19.43212)$ and $(-0.03100, -20.56340)$. As a high overtone test, the $l = 2$ value at $q = 399$ was found to be $(-199.72737, -0.11149)$. See Ref. [11] for further discussion of the method.

Validity

An important omission from our original discussions was proof that the quasinormal mode wave functions constructed from (2), with quasinormal frequencies found as the roots of the continued-fraction expression (9), completely exclude exponentially decreasing in-going behavior at spatial infinity. At the time we were content with the observation that, to within the numerical accuracy of both methods, the quasinormal frequencies computed by the continued-fraction method agreed with the fundamental and lowest overtone values computed by Chandrasekhar and Detweiler [12] by direct numerical integration of the associated Ricatti equation. Since (i) the continued-fraction expression (9) was derived without approximation from the exact series representation (2), and (ii) the zeros of this expression were numerically stable (i.e., the values of the roots obtained were both independent of the number of times the fraction was inverted, and had the expected behavior as the convergence tolerance of the fraction and root search algorithm were varied), "proof" of the validity of the higher overtone values followed by numerical induction.

In addition, the asymptotic exclusion of in-going wave behavior from the quasinormal mode wave functions constructed from series (2) was also confirmed via the independent Coulomb wave function expansion method discussed in Ref. [13]. There the Wronskian of the solution wave functions in-going at the horizon and out-going at infinity was computed in a small neighborhood of the fundamental and first five overtone quasinormal frequencies (the program has since been extended to include overtones six through ten); the Wronskian vanished at the quasinormal frequencies as required, and those lowermost six quasinormal modes were shown graphically to contribute to the complete wave form in a physically reasonable manner.

A more recent analytic continuation argument is that the wave functions constructed by the continued-fraction method obviously exclude the corresponding exponentially *increasing* behavior from the infinity of bound-state eigenfunctions of the negative inverted potential (a Stürm-Liouville problem closely related to the bound-state problem of finite dipoles [14]), and the wave function expression (2) and continued-fraction (9) are analytic functions of the potential. Therefore the analytic asymptotic functional form of the bound-state eigenfunctions should be preserved as the negative potential is reinverted to regain the Regge-Wheeler potential, and exclude the now exponentially decreasing asymptotic term. This can be demonstrated explicitly by tracking the eigenfrequency dependence as the (complex) quantity $A = l(l+1)$ is varied from positive to negative real values, keeping its magnitude constant.

DISCUSSION AND CONCLUSION

Although we have reviewed here the use of continued-fractions in determining the quasinormal frequencies and modes of Schwarzschild black-holes, the method has been shown appropriate for Kerr and Reissner-Nordström black-holes as well [4, 11]. The charged-rotating Kerr-Newman black-hole has considerable complications. The perturbation equations for the Kerr-Newman black-hole have thus far been separated only for single-spin perturbations; i.e., one of the Einstein or Maxwell fields is held fixed while the other is perturbed. This separation was effected by Dudley and Finley [15]; in the Kerr-Newman limit their radial equation (5.18b) reduces to a spheroidal wave equation, while their angular equation (5.18a) appears to be ellipsoidal for nonzero electric charge. If this is the case, the angular equation may be solved by methods similar to the matrix-determinant or continued-fraction methods discussed in [11], while the radial equation and the coupled eigenfrequency/angular separation constant problem may be treated as in the Kerr case [4]. It must be stressed, however, that the single-spin perturbations are not physically realizable [15, 16]; the actual quasinormal frequencies and modes of the Kerr-Newman black-hole will most likely be found only by solving, e.g., Bose's [17] coupled equations (23) and (24) through matrix-determinant methods such as those discussed at the conclusion of Ref. [11].

It was showed in Ref. [13] that the physical significance of high-overtone quasinormal modes can most likely be assessed only when the corresponding Wronskian derivative can also be evaluated; the Coulomb wave function expansion of the outgoing solution was found to be a suitable tool. Seidel [18] has recently modeled stellar collapse waveforms that suggest the contributions from at least the first overtone mode may be observable in addition to that from the fundamental. His results are not conclusive, however, and it would be interesting to compare his integrated waveforms in the asymptotic region with those propagated by the Green's function methods of Ref. [13] Sec. IV.

Andersson [19] has recently verified the values of the lowest ten $l = 2$ Schwarzschild gravitational quasinormal overtones by his phase-amplitude method (save for the purely imaginary eighth overtone, for which his method apparently does not apply); he should publish shortly. Among other results he finds that the quasinormal frequency values obtained via the phase-integral approximation are more accurate when the method is applied to Zerilli's potential than when applied to the Regge-Wheeler potential, even though the exact value of the quasinormal frequencies is the same for each. This may be relevant to the recently attempted application by Guinn *et al.* [2] of the JWKB method to high-order overtones. While it is perhaps premature to hope that the Anderson-Price analysis of the intertwining between these and similarly related potentials will eventually provide better insight into the applicability of the phase integral and JWKB approximations to quasinormal mode-type problems, it is difficult to imagine their article being more timely. And while it seems likely continued-fraction/matrix-determinant methods currently afford the most accurate and reliable means to compute quasinormal frequencies and modes for all physically realizable values of the black-hole parameters, there is no reason to suspect they will not be augmented (or even supplanted) by other methods in the future [20].

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